$3D$ scalar model as a $4D$ perfect conductor limit: dimensional reduction and variational boundary conditions

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Abstract

Under dimensional reduction, a system in D spacetime dimensions will not necessarily yield its $D-1$-dimensional analog version. Among other things, this result will depend on the boundary conditions and the dimension D of the system. We investigate this question for scalar and abelian gauge fields under boundary conditions that obey the symmetries of the action. We apply our findings to the Casimir piston, an ideal system for detecting boundary effects. Our investigation is not limited to extra dimensions and we show that the original piston scenario proposed in 2004, a toy model involving a scalar field in $3D$ (2+1) dimensions, can be obtained via dimensional reduction from a more realistic $4D$ electromagnetic (EM) system. We show that for perfect conductor conditions, a D-dimensional EM field reduces to a $D-1$ scalar field and not its lower-dimensional version. For Dirichlet boundary conditions, no theory is recovered under dimensional reduction and the Casimir pressure goes to zero in any dimension. This "zero Dirichlet" result is useful for understanding the EM case. We then identify two special systems where the lower-dimensional version is recovered in any dimension: systems with perfect magnetic conductor (PMC) and Neumann boundary conditions. We show that these two boundary conditions can be obtained from a variational procedure in which the action vanishes outside the bounded region. The fields are free to vary on the surface and have zero modes, which survive after dimensional reduction.

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1 Introduction

In many applications one considers the result of dimensional reduction, in which one dimension of a field theory is made small (or reduced to zero) with suitable boundary conditions imposed along this dimension. In this paper, the dimension we reduce is not "curled up", but instead is taken along an interval with boundary conditions that respect the symmetries of the original action (typically Lorentz and gauge invariance). This process of dimensional reduction is not limited to extra dimensions; it applies equally well to $4D$ $(3+1)$ systems with material boundaries, as long as they are idealized so that the symmetries of the action are obeyed. The original action is decomposed into massless and massive sectors of one lower dimension after the boundary conditions are included and one dimension is "integrated out". The effective action after dimensional reduction is then obtained. We apply these results to a particular physical system, the Casimir piston, and show that the original Casimir piston scenario introduced in 2004 [1] for a $3D$ scalar field obeying Dirichlet boundary conditions can be obtained via dimensional reduction from a $4D$ electromagnetic (EM) system obeying perfect conductor conditions. Simply put, a toy model involving a $3D$ scalar field emerges from a more realistic $4D$ EM system.

A question of general interest is whether the lower-dimensional version of a system is recovered under dimensional reduction. For example, consider a massless scalar field in $d+1$ dimensions with one dimension compactified to a circle of radius R. Sending $R \to 0$ yields its lower-dimensional version, a d-dimensional scalar field, because the Fourier decomposition of the field includes an $n=0$ mode, which yields exactly the d-dimensional scalar field. The non-zero modes become infinitely massive as $R \to 0$ and can be ignored. This scenario, however, does not apply here, since the dimension we reduce is not curled up. The boundary conditions we consider are perfect magnetic conductor (PMC) and perfect electric conductor (PEC) conditions for abelian gauge fields and Dirichlet or Neumann boundary conditions for scalar fields. In only two of these four cases does one recover the lower-dimensional version of the field under dimensional reduction.

We begin our study with massless abelian gauge (EM) fields obeying perfect electric conductor (PEC) conditions. We show that this system does not reduce to its lower-dimensional version under dimensional reduction except in $4D$. We then apply our results to the Casimir piston. Besides being an ideal system for detecting purely boundary effects, work carried out in the last four years has provided formulas for the $3+1$ Dirichlet piston [2], $3+1$ EM piston [3, 4] and higher-dimensional non-compactified scenarios [5, 6, 7]. There has also been a large amount of recent work in this field [8, 9, 10, 11, 12] (see also the introduction to [6] for some historical details). Dimensional reduction in pistons for scalar fields in Kaluza-Klein scenarios has recently been discussed in [9].

We show that Cavalcanti's original piston scenario [1], a toy model involving a $3D$ scalar

field, can be obtained via dimensional reduction from a realistic $4D$ electromagnetic system. We then investigate scalar fields under Dirichlet boundary conditions. Under dimensional reduction, no theory is recovered and the Dirichlet Casimir piston yields zero Casimir force. We show that this "zero Dirichlet" result is useful for understanding other systems like the PEC system. Perfect magnetic conductor (PMC) conditions are dual to PEC conditions and obey the same symmetries, namely Lorentz and gauge invariance. PMC conditions for EM fields and Neumann conditions for scalar fields have the same special property under dimensional reduction: they yield their lower-dimensional versions in any dimension. We explain this phenomenon by showing that they are obtained through a variational procedure in which the action vanishes outside the bounded region but with the field not fixed at the boundary surface. In both cases, the fields have zero modes, which survive after dimensional reduction.

2 Dimensional reduction of a PEC system: $4D$ EM to $3D$ scalar

Perfect electric conductor (PEC) boundary conditions can be generalized to any dimension. In a $d+1$ dimensional spacetime they are given by

$$\eta^{\mu} F^{*}_{\mu\,\alpha_1\,\alpha_2\,\alpha_{d-2}} = 0 \tag{2.1}$$

where η^μ is a spacelike vector normal to the bounded hypersurface. F^* is the dual to the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and is defined by

$$F^{*}_{\alpha_1\,\alpha_2\,\dots\,\alpha_{d-1}} \equiv \varepsilon_{\alpha_1\,\alpha_2\,\dots\,\alpha_{d-1}\,\mu\,\nu}\, F^{\mu\nu} \tag{2.2}$$

where $\varepsilon_{\alpha_1\,\alpha_2\,\alpha_{d-1}\mu\nu}$ is the $d+1$ dimensional Levi-Civita tensor. The PEC conditions (2.1) are Lorentz and gauge invariant and hence preserve the symmetries of the higher-dimensional Maxwell action. In 3+1 dimensions, they yield the familiar boundary conditions at the surface of a perfect conductor: $\mathbf{n}\times\mathbf{E} = 0$ and $\mathbf{n}\cdot\mathbf{B} = 0$, where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields respectively and $\mathbf{n}$ is the vector normal to the surface.

Consider two parallel hyperplanes situated at $x^d = 0$ and $x^d = L$ with normal vector η^μ in the x^d direction. The following mode decomposition for the gauge fields satisfy the PEC condition (2.1) at the two planes:

$$\begin{aligned}
A_\mu(x^\mu, x^d) &= \sum_{n=1}^{\infty} A_\mu^{(n)}(x^\mu)\,\sin(n\,\pi\,x^d/L) \qquad \mu = 0,1,\dots,d-1\\
A_d(x^\mu, x^d) &= \sum_{n=0}^{\infty} A_d^{(n)}(x^\mu)\,\cos(n\,\pi\,x^d/L)\\
&= A_d^{(0)}(x^\mu) + \sum_{n=1}^{\infty} A_d^{(n)}(x^\mu)\,\cos(n\,\pi\,x^d/L)\,.
\end{aligned} \tag{2.3}$$

One can to go to axial gauge $A_d = 0$ but it is more convenient to go to "almost" axial gauge [14] where $A_d = A_d^{(0)}$[1]. This can be achieved with the gauge function

$$\Lambda = \sum_{n=1}^{\infty} -\frac{L}{n\,\pi}\; A_d^{(n)}(x^\mu)\, \sin\left(n\,\pi\, x^d/L\right). \tag{2.4}$$

Note that after the gauge transformation, A_μ retains the same form. The mode decomposition in this gauge is given by

$$\begin{aligned} A_\mu(x^\mu, x^d) &= \sum_{n=1}^{\infty} A_\mu^{(n)}(x^\mu)\, \sin\left(n\,\pi\, x^d/L\right) \\ A_d &= A_d^{(0)}(x^\mu) \equiv \phi(x^\mu) \end{aligned} \tag{2.5}$$

where $\phi(x^\mu)$ represents a scalar field. Our metric signature is $(+,-,-,...,-)$ so that $A^d = -\phi$. The generalized Maxwell action in $d{+}1$ dimensions is given by

$$\begin{aligned} S &= \int -\frac{1}{4}\, F_{MN}F^{MN}\, d^{\,d+1}x & M,N &= 0,1,\ldots,d \\ &= \int -\frac{1}{4}\, F_{\mu\nu}F^{\mu\nu}\, d^{\,d+1}x + \int -\frac{1}{2}\, F_{\mu\, d}F^{\mu\, d}\, d^{\,d+1}x & \mu,\nu &= 0,1,\ldots,d-1. \end{aligned} \tag{2.6}$$

After substituting the mode decomposition (2.5) into (2.6) and integrating over x^d from 0 to L, we obtain the following action:

$$S = \frac{L}{2}\int \partial_\mu\phi\,\partial^\mu\phi\; d^{\,d}x + \frac{L}{2}\int \sum_{n=1}^{\infty}\left\{-\frac{1}{4}\, F_{\mu\nu}^{(n)}F^{\mu\nu\,(n)} + \frac{1}{2}\frac{n^2\pi^2}{L^2}\, A_\mu^{(n)}A^{\mu\,(n)}\right\} d^{\,d}x \tag{2.7}$$

where $F_{\mu\nu}^{(n)} \equiv \partial_\mu A_\nu^{(n)} - \partial_\nu A_\mu^{(n)}$. The original $d{+}1$-dimensional Maxwell action has decomposed into a massless d-dimensional massless scalar field $\phi(x^\mu)$ and an infinite tower of d-dimensional massive spin 1 fields $A_\mu^{(n)}$ of mass $m_n = n\pi/L$. Under dimensional reduction, i.e. as $L \to 0$, the spin 1 modes become infinitely massive and the theory reduces to a d-dimensional massless scalar field. Therefore, under PEC conditions, the lower-dimensional version of the original system, a d-dimensional EM field, is not recovered after dimensional reduction. There is, however, one exception. A d-dimensional EM field has $d{-}2$ degrees of freedom and therefore has one degree of freedom when $d = 3$. In other words, a $3D$ EM field and a $3D$ scalar field both have one degree of freedom and the two can be thought to be equivalent (as long as

[1] One cannot eliminate the zero mode field $A_d^{(0)}$ via a gauge transformation. If one uses axial gauge $A_d = 0$ instead of "almost" axial gauge, then $A_d^{(0)}$ appears in the new A_μ and one obtains the same action as (2.7) though it is slightly longer to derive. See appendix A.

the boundary conditions match). For PEC conditions there is no other dimension where the lower-dimensional version is recovered.

In 2004, a piston geometry was introduced for Casimir calculations [1]. The piston separates two regions, each of which contributes to the Casimir force on the piston. The original scenario was a toy model involving a $3D$ scalar field obeying Dirichlet conditions in a rectangular cavity (see Fig. 1(a)). Later, a more realistic system, the $4D$ PEC piston (see Fig. 1(b)) was solved exactly [3, 4, 6, 7]. We now show that these two systems are related: the original toy model, the $3D$ Dirichlet Casimir piston, can be viewed as a limiting case of the more realistic $4D$ PEC Casimir piston as one dimension is reduced. The Casimir force on a piston for PEC conditions in $4D$ with plate separation a and sides b and c can be expressed in many equivalent but different forms. The form found in [6, 7] is the most convenient for our purposes:

$$\begin{aligned} F_{PEC} = \frac{1}{2\,c}\sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{n}{\ell}\frac{\partial}{\partial a}K_1\Big(\frac{2\,\pi\,n\,\ell\,a}{c}\Big) \\ + \frac{\partial}{\partial a}\Bigg\{\frac{a\,c}{2}\sum_{n=1}^{\infty}\sum_{\ell_1=1}^{\infty}\sum_{\ell_2=-\infty}^{\infty}\Big(\frac{n}{b}\Big)^{3/2}\frac{K_{\frac{3}{2}}\big(\frac{2\pi n}{b}\sqrt{(\ell_1\,a)^2+(\ell_2\,c)^2}\,\big)}{[(\ell_1\,a)^2+(\ell_2\,c)^2]^{\frac{3}{4}}}\Bigg\}. \end{aligned} \tag{2.8}$$

The above formula is invariant under the exchange of b and c [6]. Without loss of generality, we choose to reduce the length b. The modified Bessel function $K_{\frac{3}{2}}\big(\frac{2\pi n}{b}\sqrt{(\ell_1\,a)^2+(\ell_2\,c)^2}\,\big)$ goes to zero exponentially as $b\to 0$. When multiplied by $1/b^{3/2}$, the product also goes to zero exponentially. Therefore the second term in (2.8) is equal to zero in the limit $b\to 0$. The result after dimensional reduction is

$$\lim_{b\to 0} F_{PEC} = \frac{1}{2\,c}\sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{n}{\ell}\frac{\partial}{\partial a}K_1\Big(\frac{2\,\pi\,n\,\ell\,a}{c}\Big) = \frac{\pi}{c^2}\sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty}n^2\,K_1^{'}\Big(\frac{2\,\pi\,n\,\ell\,a}{c}\Big) \tag{2.9}$$

where $K_1^{'}(x) = dK_1(x)/dx$. The above result (2.9) is exactly equal to the formula [eq.(11)] derived in [1] for the Casimir force on a piston for a scalar field obeying Dirichlet boundary conditions in $3D$. We have therefore shown that the $4D$ PEC piston reduces to the $3D$ Dirichlet piston under dimensional reduction. It would be interesting to see if high-precision Casimir experiments [16] with real metals in $4D$ can be developed in the near future to verify this "reduction" scenario.

We end this section by clarifying a point. As already stated, a $3D$ EM field and a $3D$ scalar field each have one degree of freedom. For free fields the two are automatically equivalent. However, if we impose boundary conditions on the EM field, we need to find the corresponding boundary conditions on the scalar field, which in general will not be solely Dirichlet nor Neumann boundary conditions. Let us look at a concrete example. Consider a $3D$ EM field confined to an $L_1 \times L_2$ rectangular region with PEC boundary conditions. In radiation gauge ($A_0 =$

$0, \nabla \cdot \mathbf{A} = \mathbf{0})$, PEC conditions yield $A_{||} = 0$ and $\partial_n A_\perp = 0$ (where $||$ means parallel to the surface and ∂/∂_n denotes the normal derivative). The mode decomposition is given by

$$\begin{aligned} &A_0 = 0 \\ &A_1 = a_{n_1 n_2} \cos(n_1 \pi x_1/L_1) \sin(n_2 \pi x_2/L_2) \qquad\qquad n_1, n_2 \geq 0\ ;\ \ (n_1, n_2) \neq (0,0). \\ &A_2 = b_{n_1 n_2} \cos(n_2 \pi x_2/L_2) \sin(n_1 \pi x_1/L_1)\,. \end{aligned}$$

If $n_1 = 0$ and $n_2 \neq 0$, then $A_2 = 0$ and $A_1 \neq 0$ and if $n_2 = 0$ and $n_1 \neq 0$, then $A_1 = 0$ and $A_2 \neq 0$. When n_1 and n_2 are both positive, the condition $\nabla \cdot \mathbf{A} = \mathbf{0}$ yields a relation between the two coefficients:

$$b_{\,n_1 n_2} = -\frac{n_1\, L_2}{n_2\, L_1}\, a_{\,n_1 n_2} \qquad n_1, n_2 \in \mathbb{Z}+\,. \tag{2.10}$$

The 3D EM field under PEC conditions is therefore equivalent to a $3D$ scalar field given by :

$$\phi_{n_1 n_2} = a_{n_1 n_2} \cos(n_1 \pi x_1/L_1) \sin(n_2 \pi x_2/L_2) + b_{n_1 n_2} \cos(n_2 \pi x_2/L_2) \sin(n_1 \pi x_1/L_1) \tag{2.11}$$

where $n_1, n_2 \geq 0, (n_1, n_2) \neq (0,0)$ and $b_{n_1 n_2}$ is related to $a_{n_1 n_2}$ via (2.10) when n_1 and n_2 are both positive. The above scalar field satisfies the boundary conditions imposed on the EM field but they do not correspond to either Dirichlet or Neumann boundary conditions on an $L_1 \times L_2$ rectangular region. However, the Casimir energy is equal to the Neumann Casimir energy since the frequency ω for a given mode is given by $(\frac{n_1^2 \pi^2}{L_1^2} + \frac{n_2^2 \pi^2}{L_2^2})^{1/2}$ and the sum is over the same modes (n_1, n_2) (except for the mode $(0,0)$ that appears in the Neumann case but makes no contribution to the Casimir energy).

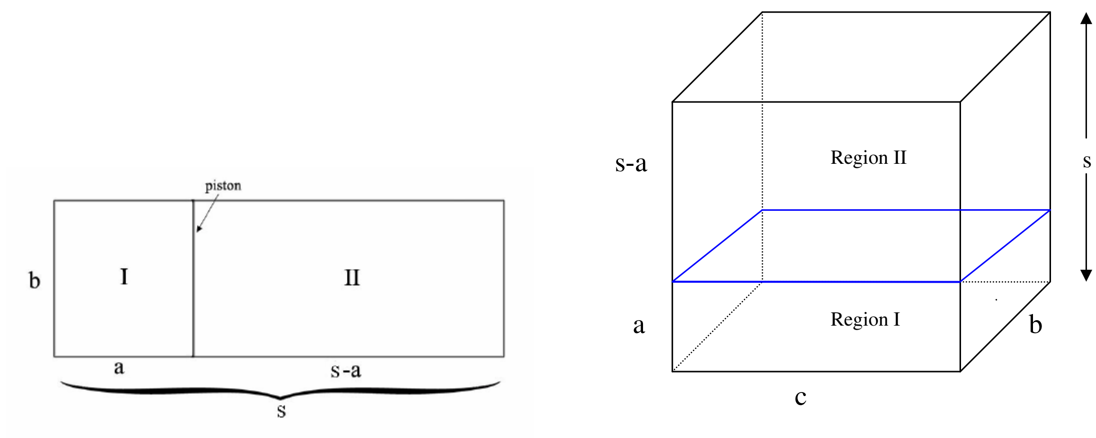
3 No theory recovered under Dirichlet and PEC revisited

We now consider a $d{+}1$ dimensional massless scalar field ϕ obeying Dirichlet boundary conditions at the hyperplanes $x^d = 0$ and $x^d = L$. Because the electromagnetic case with PEC and PMC conditions can be understood as sums over Dirichlet systems [6, 7], this case forms a foundation for understanding more complicated systems. The Fourier decomposition of the scalar field under Dirichlet conditions is given by

$$\phi(x^\mu, x^d) = \sum_{n=1}^{\infty} \phi^n(x^\mu) \sin(n\, \pi\, x^d/L) \quad \mu = 0, 1, 2, \ldots, d-1\ . \tag{3.12}$$

After integrating over x^d from 0 to L, we can express the action for the scalar field as

$$\begin{aligned} S &= \int \frac{1}{2}\, \partial_M \phi\, \partial^M \phi\ \ d^{d+1}x \qquad M = 0, 1, 2, ..., d \\ &= \frac{L}{2} \sum_{n=1}^{\infty} \int \Big(\frac{1}{2}\, \partial_\mu \phi^n\, \partial^\mu \phi^n + \frac{n^2\, \pi^2}{L^2} \, (\phi^n)^2 \Big)\ d^{\,d}x\,. \end{aligned} \tag{3.13}$$



(a) Piston geometry in two spatial dimensions. The length s is taken to be infinite, a is the plate separation and b is the length of the second side. The $3D$ Dirichlet piston is a scalar field obeying Dirichlet boundary conditions on all sides.

(b) Piston geometry in three spatial dimensions. The plate separation is a and s is taken to be infinite. The $4D$ PEC piston is an EM field obeying perfectly conducting conditions on all walls. In the limit as one of the sides (b or c) goes to zero, we recover the $3D$ Dirichlet piston.

Figure 1: piston geometry

The massless scalar field in $d+1$ dimensions decomposes into an infinite tower of massive scalar fields in d dimensions. Note that the sum starts at $n = 1$. Under dimensional reduction, i.e. as $L \to 0$, every term becomes infinitely massive and one does not recover any theory. Correspondingly, the Dirichlet Casimir piston yields a zero Casimir energy as one dimension is reduced to zero. The Casimir energy for the Dirichlet Casimir piston in $d+1$ dimensions is given by [13]:

$$E_D = \frac{-1}{2\pi} \sum_{\{n_i\}=1}^{\infty} \sum_{\ell=1}^{\infty} \frac{\lambda}{\ell}\, K_1(2\,\ell\,\lambda\,a) \quad \text{where } \lambda = \left[\sum_{i=1}^{d-1} \frac{\pi^2\, n_i^2}{L_i^2} \right]^{1/2} . \tag{3.14}$$

Here a is the plate separation and the L_i are the lengths of the remaining $d-1$ sides. The sum over each n_i starts at 1. We are interested in the limit as one of the lengths tends to zero, so without loss of generality we choose to reduce L_{d-1}. As $L_{d-1} \to 0$, $\lambda \to \infty$ and the modified Bessel function $K_1(2\,\ell\,\lambda\,a)$ goes to zero exponentially, yielding a zero Casimir energy:

$$\lim_{L_{d-1}\to 0} E_D = 0\,. \tag{3.15}$$

The Dirichlet Casimir energy is zero under dimensional reduction. We will refer to this as the "zero Dirichlet" result.

In the previous section, we showed that the $4D$ PEC piston reduces to the $3D$ Dirichlet piston under dimensional reduction. We are now in a position to show this using the "zero Dirichlet" result. The $4D$ PEC piston can be decomposed into sums over Dirichlet pistons of different dimensions [6, 7]:

$$E_{PEC_{123}} = 2\, E_{D_{123}} + E_{D_{12}} + E_{D_{13}} + E_{D_{23}}\,. \tag{3.16}$$

Here $E_{PEC_{123}}$ and $E_{D_{123}}$ represent the $4D$ PEC and Dirichlet energies respectively in a rectangular geometry with three sides of length L_1, L_2 and L_3, and $E_{D_{12}}$ represents the $3D$ Dirichlet energy with two sides of length L_1 and L_2. We can dimensionally reduce the $4D$ PEC piston by letting $L_3 \to 0$. The "zero Dirichlet" result implies that the Dirichlet Casimir energies containing the length L_3 go to zero as $L_3 \to 0$. We therefore obtain $\lim_{L_3\to 0} E_{PEC_{123}} = E_{D_{12}}$ which is the $3D$ Dirichlet piston with lengths L_1 and L_2.

4 Lower-dimensional system recovered: PMC and Neuman conditions

Perfect magnetic conductor (PMC) boundary conditions are dual to perfect electric conductor (PEC) conditions. They are given in any dimension by $\eta^\mu\, F_{\mu\nu} = 0$, where η^μ is a spacelike vector normal to the hypersurface and $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor [6]. PMC conditions obey the symmetries of the Maxwell action, namely Lorentz and gauge invariance. In 3+1 dimensions, the conditions at the surface reduce to $\mathbf{n}\cdot\mathbf{E} = 0$ and $\mathbf{n}\times\mathbf{B} = 0$, where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields and $\mathbf{n}$ is the vector normal to the surface. Material structures that approximate PMC's are of current interest because of their usefulness to communication technologies, in particular as low-profile antennas [15]. An important property of a PMC is that its surface reflects electromagnetic waves without phase change of the electric field, in contrast to the π phase change from a PEC [15].

Consider a d+1-dimensional electromagnetic field obeying PMC conditions on two hyperplanes situated at $x^d = 0$ and $x^d = L$. The mode decomposition for the gauge fields is given by [6]

$$\begin{aligned}
A_\mu(x^\mu, x^d) &= \sum_{n=0}^{\infty} A_\mu^{(n)}(x^\mu)\, \cos\left(n\,\pi\, x^d/L\right)\\
&= A_\mu^{(0)}(x^\mu) + \sum_{n=1}^{\infty} A_\mu^{(n)}(x^\mu)\, \cos\left(n\,\pi\, x^d/L\right) \qquad \mu = 0, 1, \ldots, d-1\\
A_d(x^\mu, x^d) &= \sum_{n=1}^{\infty} A_d^{(n)}(x^\mu)\, \sin\left(n\,\pi\, x^d/L\right).
\end{aligned} \tag{4.17}$$

It is convenient to go over to axial gauge $A_d = 0$. This can be accomplished with the gauge

function

$$\Lambda = \sum_{n=1}^{\infty} \frac{L}{n\,\pi}\; A_d^{(n)}(x^\mu)\,\cos{(n\,\pi\,x^d/L)}\,, \tag{4.18}$$

which does not affect the form of A_μ. In axial gauge, the mode decomposition becomes

$$\begin{aligned} A_\mu &= A_\mu^{(0)}(x^\mu) + \sum_{n=1}^{\infty} A_\mu^{(n)}(x^\mu)\,\cos{(n\,\pi\,x^d/L)} \\ A_d &= 0\,. \end{aligned} \tag{4.19}$$

We now substitute the mode decomposition (4.19) into the $d{+}1$-dimensional Maxwell action (2.6). After integrating over x^d from 0 to L, the resulting action is:

$$S = \frac{L}{2}\int -\frac{1}{4}\,F_{\mu\nu}^{(0)}F^{\mu\nu\,(0)}\,d^{\,d}x \;+\; \frac{L}{2}\int\sum_{n=1}^{\infty}\Big\{-\frac{1}{4}\,F_{\mu\nu}^{(n)}F^{\mu\nu\,(n)} + \frac{1}{2}\frac{n^2\pi^2}{L^2}\,A_\mu^{(n)}A^{\mu\,(n)}\Big\}\,d^{\,d}x \tag{4.20}$$

where $F_{\mu\nu}^{(0)} \equiv \partial_\mu A_\nu^{(0)} - \partial_\nu A_\mu^{(0)}$. Under PMC conditions, the Maxwell action in $d{+}1$ dimensions decomposes into two sectors: Maxwell in d dimensions plus an infinite tower of d-dimensional massive gauge fields. Under dimensional reduction, where we let $L \to 0$, the mass of the gauge fields becomes infinite so that the massive sector can be ignored. Therefore, for any starting dimension, we recover the lower-dimensional version of the original system: the d-dimensional Maxwell action. In contrast, for PEC conditions, the lower-dimensional version was recovered only in $4D$ and for Dirichlet boundary conditions no theory is recovered at all.

We now show that the PMC Casimir piston in $d{+}1$ dimensions reduces to the d-dimensional PMC piston under dimensional reduction. For this purpose, a convenient formula for the Casimir force on a PMC piston in $d{+}1$ dimensions is expression $A.6$ found in [6]:

$$F_{PMC} = -\sum_{p=1}^{d-1}\sum_{q=0}^{d-p-1}\frac{\pi}{2^{d-q+1}}\,(d{-}1{-}2p{-}q)\;\xi^{\,d-q-1}_{\,1,k_2,k_3,..,k_p}\,\frac{a_{k_2-1}\cdots a_{k_p-1}}{(a_{d-q-1})^{p+1}}\,\frac{\partial}{\partial\,a}\big\{a\,R_p\big\} \tag{4.21}$$

where

$$R_p = \sum_{n=1}^{\infty}\sum_{\ell_1=1}^{\infty}\sum_{\substack{\ell_i=-\infty\\ i=2,\ldots,p}}^{\infty}\frac{4\;n^{\frac{p+1}{2}}}{\pi}\;\frac{K_{\frac{p+1}{2}}\Big(2\pi\,n\,\sqrt{(\ell_1\frac{a}{a_{d-q-1}})^2+\cdots+(\ell_p\,\frac{a_{k_p-1}}{a_{d-q-1}})^2}\;\Big)}{\Big[(\ell_1\frac{a}{a_{d-q-1}})^2+\cdots+(\ell_p\frac{a_{k_p-1}}{a_{d-q-1}})^2\Big]^{\frac{p+1}{4}}}\,. \tag{4.22}$$

In the above formula, a is the plate separation and the lengths of the other $d{-}1$ sides are $a_1, a_2, \ldots, a_{d-1}$. We are interested in evaluating the limit of F_{PMC} as one of the lengths is reduced to zero. Without loss of generality we choose the length to be a_{d-1}. This length

appears in (4.21) when $q=0$ in a_{d-q-1}, i.e. it appears in the denominator in (4.21) and in the argument of the modified Bessel function of R_p in (4.22) when $q=0$. We are therefore interested in taking the limit of only the $q=0$ terms. In the limit $a_{d-1}\to 0$ it is easy to see that those terms are zero, because the modified Bessel functions that appear in R_p go to zero exponentially in this limit. We obtain[2] $\lim_{a_{d-1}\to 0}(a_{d-1})^{-p-1}\,\partial\,(a\,R_p)/\partial\,a\,=0$. After taking the limit, the sum over p now runs from 1 to $d-2$ instead of $d-1$. The sum over q runs from 1 to $d-p-1$. This is equivalent to q running from 0 to $d-p-2$ if we replace d by $d-1$ in the summand of (4.21). The final result is thus an identical formula to F_{PMC}, except that d is now replaced by $d-1$:

$$\lim_{a_{d-1}\to 0} F_{PMC} = -\sum_{p=1}^{d-2}\sum_{q=0}^{d-p-2}\frac{\pi}{2^{d-q}}\,(d-2-2p-q)\;\xi^{\,d-q-2}_{\,1,k_2,k_3,..,k_p}\,\frac{a_{k_2-1}\dots a_{k_p-1}}{(a_{d-q-2})^{p+1}}\frac{\partial}{\partial\,a}\big\{a\,R_p\big\}. \tag{4.23}$$

where R_p is given by (4.22) with d replaced by d–1. We therefore recover the lower-dimensional version of the PMC Casimir force under dimensional reduction.

We now consider a d+1 dimensional massless scalar field ϕ obeying Neumann boundary conditions $\eta^\mu\partial_\mu\phi=0$ at the hyperplanes $x^d=0$ and $x^d=L$, where η^μ is the spacelike vector in the x^d direction normal to the hyperplanes. These conditions are Lorentz invariant. The Fourier decomposition of the scalar field under Neumann conditions is

$$\begin{aligned}\phi(x^\mu,x^d) &= \sum_{n=0}^{\infty}\phi^n(x^\mu)\,\cos(n\,\pi\,x^d/L)\qquad \mu=0,1,2,\dots,d-1\\ &= \phi^{(0)}(x^\mu)+\sum_{n=1}^{\infty}\phi^n(x^\mu)\,\cos(n\,\pi\,x^d/L)\,.\end{aligned} \tag{4.24}$$

After integrating over x^d from 0 to L the action for the scalar field can be expressed as

$$\begin{aligned}S &= \int\frac{1}{2}\,\partial_M\phi\,\partial^M\phi\;\,d^{\,d+1}x\qquad M=0,1,2,...,d\\ &= \frac{L}{2}\,\int\frac{1}{2}\,\partial_\mu\phi^{(0)}\,\partial^\mu\phi^{(0)}\,d^{\,d}x+\frac{L}{2}\,\sum_{n=1}^{\infty}\int\Big(\frac{1}{2}\,\partial_\mu\phi^n\,\partial^\mu\phi^n+\frac{n^2\,\pi^2}{L^2}\,(\phi^n)^2\Big)\;d^{\,d}x\,.\end{aligned} \tag{4.25}$$

The massless scalar field in d+1 dimensions under Neumann conditions has decomposed into a massless sector with a scalar field in d dimensions plus a massive sector with an infinite tower of d-dimensional massive scalar fields $\phi^{(n)}$ of mass $m_n=n\,\pi/L$. Under dimensional reduction, $L\to 0$, the masses go to infinity, so that the massive sector can be ignored. We therefore

[2]The ordered symbol $\xi^{\,d-q-1}_{\,1,k_2,k_3,..,k_p}$ introduced in [2] and the product $a_{k_2-1}\dots a_{k_p-1}$ do not contain a_{d-1}.

recover the lower-dimensional version of the original system: a d-dimensional massless scalar field.

We have encountered two special systems, the PMC and Neumann systems, where the lower-dimensional version of the original system in any dimension is recovered under dimensional reduction. What do these boundary conditions have in common? They both are "variational" conditions that can be obtained by minimizing the action with the field free to vary on the surface. Such conditions arise naturally in bag models [17].

We vary the action $S[\Phi]$ with respect to the field Φ, where $S[\Phi]$ vanishes outside a bounded region. Assuming the equations of motion are satisfied, we obtain the following boundary term (which must be set to zero)

$$\int \partial_\mu \Big(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \Phi)} \, \delta\Phi \Big) \, d^{d+1}x = \int \eta_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \Phi)} \, \delta\Phi \; d\,\sigma = 0 \tag{4.26}$$

where η^μ is a spacelike vector normal to the timelike hypersurface σ. If Φ is allowed to vary on the boundary[3] i.e. $\delta\Phi \neq 0$, we then obtain the following Lorentz invariant boundary condition:

$$\eta_\mu \, \frac{\partial \mathscr{L}}{\partial(\partial_\mu \Phi)} = 0\,. \tag{4.27}$$

If Φ is a Klein-Gordon scalar field ϕ, then $\mathscr{L} = \frac{1}{2}\, \partial_\mu \phi \, \partial^\mu \phi$ and we obtain Neumann boundary conditions $\eta^\mu \partial_\mu \phi = 0$, and if Φ is an abelian gauge field A_ν, then $\mathscr{L} = -\frac{1}{4}\, F_{\mu\nu} F^{\mu\nu}$ and we obtain PMC conditions $\eta^\mu F_{\mu\nu} = 0$. In both cases, there is no momentum flux through the hypersurface σ even though the field is free to vary on the surface. For the scalar field, $\eta_\mu \, T^{\mu 0} = (\eta_\mu \, \partial^\mu \phi)\, \partial^0 \phi$, and this is equal to zero for Neumann boundary conditions. For an EM field, $\eta_\mu \, T^{\mu 0} = (\eta_\mu \, F^{\mu\alpha})\, F^0{}_\alpha$, and this is equal to zero for PMC conditions.

For PMC or Neumann boundary conditions, where the fields are free to vary on the surface, the fields have zero modes. These zero modes survive after dimensional reduction and one recovers the lower-dimensional version of the original action. This would also occur in chiral models, since the Dirac field would not be fixed on the boundary and would have zero modes. This is in contrast to Dirichlet and PEC conditions. For Dirichlet, the scalar field has no zero modes and under dimensional reduction it becomes infinitely massive. PEC conditions are more interesting. One does not recover the lower-dimensional Maxwell action under dimensional reduction but the action of a scalar field. Only one component of the gauge fields has a zero mode.

[3] Dirichlet and PEC boundary conditions do not fulfill this criteria. For the case of Dirichlet, the field ϕ is zero on the boundary and for the PEC case, the gauge fields A_ν are zero on the boundary except for one component.

5 Conclusion

We have studied dimensional reduction for abelian gauge fields under PMC and PEC conditions and for scalar fields under Dirichlet and Neumann boundary conditions. Our investigation was not restricted to extra dimensions and included dimensional reduction of $4D$ systems. In particular, PEC and PMC conditions can be viewed as idealized material boundary conditions for EM fields in 4D. We showed that for PEC conditions, a D-dimensional EM field reduces under dimensional reduction to a $D-1$ scalar field, and not its lower-dimensional version, a $D-1$ EM field. In particular, we showed that the $3D$ Dirichlet piston can be obtained via dimensional reduction from a more realistic $4D$ EM piston obeying perfect conductor conditions. While the $3D$ scalar field system was a toy model and inaccessible experimentally, there is now the possibility that high-precision Casimir experiments [16] involving real metals in $4D$ could be developed to test this scenario or something similar to it. We noted that a $3D$ EM field and a $3D$ scalar field both have one degree of freedom, but are equivalent only as long as the scalar field takes into account the boundary conditions imposed on the EM field. In particular, a $3D$ EM field confined to a rectangular geometry under PEC conditions is equivalent to the $3D$ scalar field given by (2.11), which does not correspond to Dirichlet or Neumann boundary conditions even though the mode frequencies are equal to those of Neumann boundary conditions.

For Dirichlet boundary conditions, we found that under dimensional reduction no theory is recovered and the Casimir force on a Dirichlet piston is zero. This "zero Dirichlet" result is particularly useful in showing that the $4D$ PEC Casimir piston reduces to the $3D$ Dirichlet Casimir piston under dimensional reduction. We identified two special boundary conditions, perfect magnetic conductor (PMC) conditions for EM fields and Neumann conditions for scalar fields, where dimensional reduction yields the lower-dimensional version of the action in any dimension and verified explicitly this result for the PMC Casimir piston. These two cases represent "variational" boundary conditions where the action vanishes outside a bounded region and the field is not fixed on the surface. As a consequence, PMC and Neumann conditions have zero modes and these yield the lower-dimensional version of the action under dimensional reduction.

A Action for PEC system using axial gauge $A_d = 0$

In this appendix, using axial gauge $A_d = 0$, we derive the action (2.7) starting with the mode decomposition (2.3) for the EM field under PEC conditions. For axial gauge $A_d = 0$, the gauge

function (2.4) has to be modified to

$$\Lambda = \sum_{n=1}^{\infty} -\frac{L}{n\,\pi}\, A_d^{(n)}(x^\mu)\, \sin\left(n\,\pi\,x^d/L\right) - A_d^{(0)}(x^\mu)\, x^d\,. \tag{A.1}$$

A gauge transformation with the above gauge function yields the following mode decomposition for the gauge fields

$$\begin{aligned} A_\mu(x^\mu, x^d) &= \sum_{n=1}^{\infty} A_\mu^{(n)}(x^\mu)\, \sin\left(n\,\pi\,x^d/L\right) - \partial_\mu \phi\, x^d \\ A_d &= 0 \end{aligned} \tag{A.2}$$

where the first term in (A.1) has been absorbed into a redefinition of $A_\mu^{(n)}(x^\mu)$ and $\phi(x^\mu) \equiv A_d^{(0)}(x^\mu)$ represents a scalar field. We need to evaluate the Maxwell action (2.6) with the mode decomposition (A.2). The tensor $F_{\mu\nu}$ is given by

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu = \sum_{n=1}^{\infty} \left(\partial_\mu A_\nu^{(n)} - \partial_\nu A_\mu^{(n)}\right) \sin(n\,\pi x^d/L) = \sum_{n=1}^{\infty} F_{\mu\nu}^{(n)} \sin(n\,\pi x^d/L)\,. \tag{A.3}$$

The scalar field ϕ does not appear in $F_{\mu\nu}$ because the partial derivatives commute i.e. $(-\partial_\mu\partial_\nu + \partial_\nu\partial_\mu)\,\phi = 0$. Squaring the tensor and integrating over the d+1-dimensional spacetime yields

$$\int F_{\mu\nu}\, F^{\mu\nu}\, d^{\,d+1}x = \frac{L}{2} \int \sum_{n=1}^{\infty} F_{\mu\nu}^{(n)} F^{\mu\nu(n)}\, d^{\,d}x \tag{A.4}$$

where we have integrated over x^d using the orthogonal property

$$\int_0^L \sin(n\,\pi x^d/L)\, \sin(m\,\pi x^d/L)\, dx^d = \frac{L}{2}\, \delta_{nm} \qquad n, m \geq 1. \tag{A.5}$$

The tensor $F_{\mu\, d}$ is given by

$$F_{\mu\, d} \equiv \partial_\mu A_d - \partial_d\, A_\mu = -\partial_d\, A_\mu = \sum_{n=1}^{\infty} -\frac{n\pi}{L} A_\mu^{(n)} \cos(n\,\pi x^d/L) + \partial_\mu \phi\,. \tag{A.6}$$

Again, squaring the tensor and integrating yields

$$\int F_{\mu\, d}\, F^{\mu\, d}\, d^{\,d+1}x = -\frac{L}{2} \int \sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 A_\mu^{(n)} A^{\mu^{(n)}}\, d^{d}x - L \int \partial_\mu \phi\, \partial^\mu \phi\, d^{\,d}x \tag{A.7}$$

where $F^{\mu\, d} = -\partial^d A^\mu = \partial_d A^\mu$ and we have integrated over x^d using the following results:

$$\int_0^L \cos(n\,\pi x^d/L)\,\cos(m\,\pi x^d/L)\,dx^d = \frac{L}{2}\,\delta_{nm} \quad ; \quad \int_0^L \cos(n\,\pi x^d/L)\,dx^d = 0 \quad n,m \geq 1. \tag{A.8}$$

With (A.4) and (A.7), the $d+1$-dimensional Maxwell action (2.6) can be expressed as

$$\begin{aligned} S &= \int -\frac{1}{4}\,F_{MN}F^{MN}\,d^{\,d+1}x && M,N = 0,1,\ldots,d \\ &= \int -\frac{1}{4}\,F_{\mu\nu}F^{\mu\nu}\,d^{\,d+1}x + \int -\frac{1}{2}\,F_{\mu\,d}F^{\mu\,d}\,d^{\,d+1}x && \mu,\nu = 0,1,\ldots,d-1 \\ &= \frac{L}{2}\int \partial_\mu\phi\,\partial^\mu\phi\;d^{\,d}x + \frac{L}{2}\int\sum_{n=1}^{\infty}\left\{-\frac{1}{4}\,F_{\mu\nu}^{(n)}F^{\mu\nu(n)} + \frac{1}{2}\frac{n^2\pi^2}{L^2}\,A_\mu^{(n)}A^{\mu(n)}\right\}d^{\,d}x \end{aligned} \tag{A.9}$$

which is equal to the action (2.7).

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